Math 10A with Professor Stankova
Worksheet, Discussion #24; Friday, 10/20/2017
GSI name: Roy Zhao

Differential Equations

Example

1. A population model is given by $\frac{dP}{dt} = -10P(P-1)(P-5)^2$. How does the fate of the population depend on the initial population?

> **Solution:** First we find the equilibrium solutions, or when $\frac{dP}{dt} = 0$. The right side is conveniently already factored, so it is $0, 1, 5$. Then, we determine the sign of the differential in between these equilibrium solutions. At $(0, 1)$, it is increasing, at $(1, 5)$ it is decreasing, and at $(5, \infty)$ it is also decreasing. This means if you start at a population greater than 5, you go down to 5, if you have between 1 and 5, you go down to 1, if you have between 0 and 1, you go up to 1. So 0 is unstable, 1 is stable, and 5 is semistable.

Problems

2. True **FALSE** Given a differential equation $\frac{dP}{dt} = f(P)$, if we cannot solve for P, then we have no hope of describing P's behavior.

> **Solution:** This is false because finding the zeros of $f(P)$ allows us to see where the equilibria solutions are and whether they are stable/semistable/unstable which gives us a general idea of what solutions look like.

3. True **FALSE** If the rate of change of x is proportional to x, then $\frac{dx}{dt} = ax + b$.

> **Solution:** There is no extra b factor.

4. **TRUE** False The doubling time of the differential equation $\frac{dN}{dt} = 2t$ depends on the initial value of N.

Solution: Notice that the function is now $2t$ so solving gives $N = t^2 + C$. So, the doubling time will depend on the initial condition which will give the constant C.

5. Write a differential equation describing the fact that the rate of change of the mass M of an object is proportional to its cube root.

Solution:

$$\frac{dM}{dt} = k\sqrt[3]{M}.$$

6. If $\dfrac{dN}{dt} = (-\ln 2)N$, then does the doubling time or halving time make more sense? Find it.

Solution: The halving time makes sense because the population is decreasing. The formula is $\dfrac{\ln 2}{r}$, where r is your rate of growth, so the halving time is just 1.

7. A population model is given by $\dfrac{dP}{dt} = (P-1)(P-10)$. How does the fate of the population depend on the initial population?

Solution: The equilibrium solutions are $P = 1, 10$. Looking at signs, $P = 1$ is a stable solution and $P = 10$ is an unstable solution.

Separable Equations

Example

8. Find the solution to $\frac{dy}{dt} = 3t^2y^3 + e^ty^3$ with $y(0) = -1$.

Solution: We factor the right side as $y^3(t^2 + e^t)$ so we have that

$$\frac{dy}{y^3} = (3t^2 + e^t)dt.$$

Taking the integral of both sides gives

$$\frac{-1}{2y^2} = t^3 + e^t + C \implies y^2 = \frac{-1}{2t^3 + 2e^t + C}.$$

Now plugging in the initial condition that $y(0) = -1$, we have that $1 = \dfrac{-1}{0 + 2e^0 + C}$ so $2 + C = -1$ and $C = -3$. Therefore, we have

$$y^2 = \frac{-1}{2t^3 + 2e^t - 3}.$$

If we want this in terms of y only, then we have to choose which sign of y to take. We are given that $y(0) = -1$, so we take the negative square root and

$$y = -\sqrt{\frac{-1}{2t^3 + 2e^t - 3}}.$$

Problems

9. True **FALSE** We can always use the method of separable equations to solve $\dfrac{dy}{dt} = f(y,t)$.

Solution: This is false because if we cannot separate this equation, then we cannot use separable equations.

10. True **FALSE** The differential equation $\dfrac{dy}{dt} = y + t$ is separable.

Solution: A separable equation is an equation where we can split it up into a function of y **times** a function of t. Addition will not work.

11. Find the solution to $\dfrac{dy}{dt} = te^y$ with $y(0) = 1$.

Solution: Separating gives

$$dye^{-y} = tdt \implies -e^{-y} = \frac{t^2}{2} + C.$$

Taking a negative sign and ln gives $-y = \ln(-t^2/2 - C)$ so $y = -\ln(-t^2/2 - C)$. Now we have that $y(0) = 1$ so $1 = -\ln(-0 - C)$ so $-1 = \ln(-C)$ and $-C = e^{-1}$ and $C = -e^{-1}$. So our function is

$$y = -\ln(-t^2/2 + e^{-1}).$$

12. Find the solution to $\dfrac{dy}{dx} = 6xy^2$ with $y(1) = 1/4$.

Solution: Splitting gives

$$\frac{dy}{y^2} = 6xdx \implies \frac{-1}{y} = 3x^2 + C \implies y = \frac{-1}{3x^2 + C}.$$

Plugging in $y(1) = \frac{1}{4}$ gives $\frac{1}{4} = \frac{-1}{3+C}$ so $3 + C = -4$ and $C = -7$. Therefore, we have

$$y = \frac{-1}{3x^2 - 7}.$$

13. Find the solution to $\dfrac{dy}{dx} = \frac{3x^2+2x+1}{2y+1}$ with $y(0) = 1$.

Solution: Split and get

$$(2y + 1)dy = (3x^2 + 2x + 1)dx \implies y^2 + y = x^3 + x^2 + x + C.$$

We cannot explicitly solve for y easily so we have to leave it in this form. But, we can solve for c to get $y(0) = 1$ so $1 + 1 = C = 2$. Therefore, the implicit solution is

$$y^2 + y = x^3 + x^2 + x + 2.$$

14. Find the solution to $\dfrac{dr}{dt} = \frac{r^2}{t}$ with $r(1) = 1$.

Solution: Split to get $\frac{dr}{r^2} = \frac{dt}{t}$ so $-1/r = \ln t + C$ and $r = \dfrac{-1}{\ln t + C}$. Plugging in $r(1) = 1$, we have that $1 = \dfrac{-1}{0 + C}$ so $C = -1$. Therefore, we have

$$r = \frac{-1}{\ln t - 1}.$$

15. Find the solution to $\dfrac{dy}{dt} = 2y + 3$ with $y(0) = 0$.

Solution: Separate to get

$$\frac{dy}{2y+3} = dt \implies \frac{\ln(2y+3)}{2} = t + C,$$

so $2y + 3 = e^{2t+C}$. We can bring the constant down as multiplying by a constant so $e^{2t+C} = e^{2t} \cdot e^{C} = Ce^{2t}$ and so

$$y = \frac{Ce^{2t} - 3}{2}.$$

Taking $y(0) = 0$ gives $0 = (C-3)/2$ so $C = 3$ and we get

$$y = \frac{3e^{2t} - 3}{2}.$$

16. Find the solution to $\dfrac{dx}{dy} = e^{x-y}$ with $x(0) = 0$.

Solution: We can split e^{x-y} as $e^{x} \cdot e^{-y}$ and now splitting gives

$$e^{-x}dx = e^{-y}dy \implies -e^{-x} = -e^{-y} + C$$

so $e^{-x} = e^{-y} + C$ and $-x = \ln(e^{-y} + C)$ and $x = -\ln(e^{-y} + C)$. Plugging in $x(0) = 0$ gives $0 = -\ln(1 + C)$ so $1 + C = 0$ and $C = -1$. So

$$x = -\ln(e^{-y} - 1).$$